[Very Quick] Introduction to Computational Fluid Dynamics Theory

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National Institute of Aerospace
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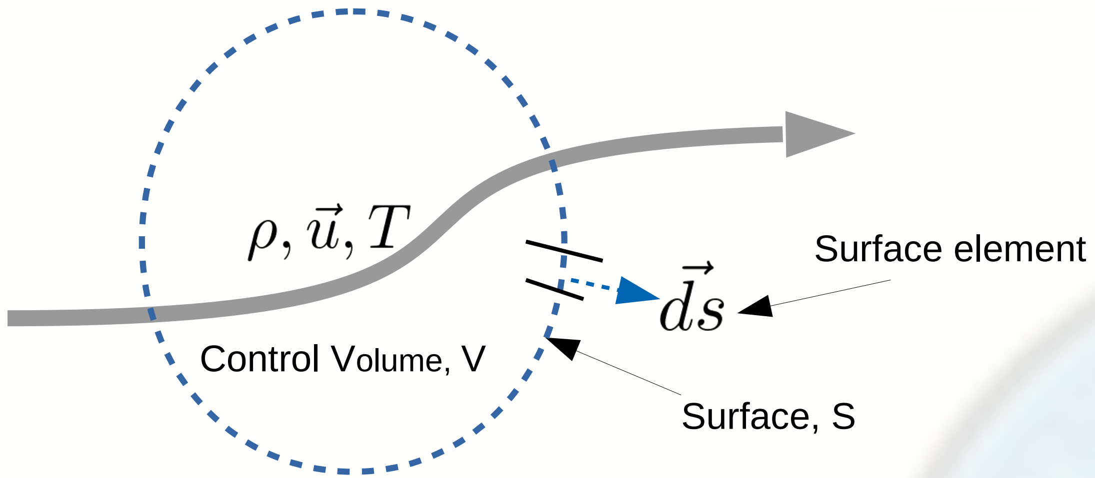
How much CFD theory can you cover in 20 minutes?

- This presentation will **NOT**:
  - Cover everything you need to know to be an expert in CFD.
  - Replace textbooks, graduate level classes, and years of industry experience.
- This presentation WILL:
  - Introduce the some of the theory behind CFD & numerical settings.
  - Provide some vocabulary to help you understand the information available in textbooks, online, and in coursework.
  - Help you troubleshoot logically.
  - Make CFD more than just a ‘black box’.

Outline

- Equations of Fluid Motion
- Numerical Methods
  - The CFL number
  - Related Vocabulary
- Conclusions & Further Reading

Equations of Fluid Motion



Conservation Equations: conservation law/divergence law form

Mass/continuity: $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Momentum: (x-direction) $$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + \sigma_x)}{\partial x} + \frac{\partial (\rho u v + \tau_{xy})}{\partial y} + \frac{\partial (\rho u w + \tau_{xz})}{\partial z} = 0$$

Energy: $$\frac{\partial e}{\partial t} + \frac{\partial ((e + \sigma_x) u + v \tau_{yx} + w \tau_{zx} - k \frac{\partial T}{\partial x})}{\partial x} + \ldots = 0$$

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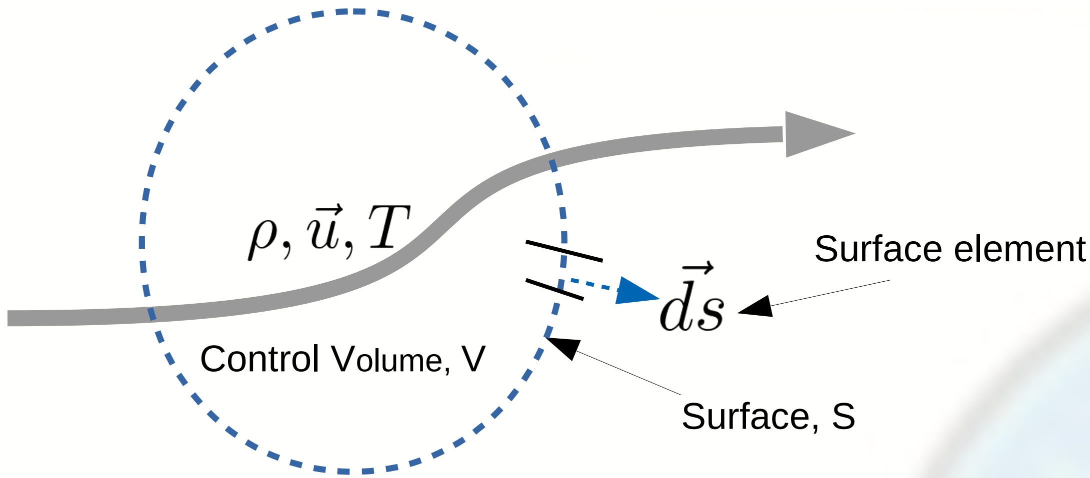
Equations of Fluid Motion



Viscous stresses & pressure

Conservation Equations: conservation law/divergence law form

Thermal conductivity

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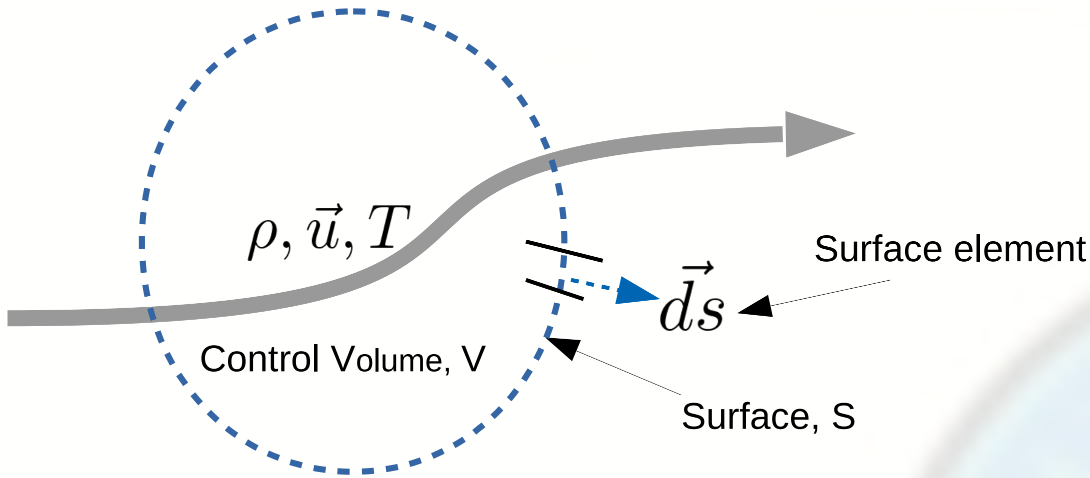
Total energy per unit volume, e(T)

Equations of Fluid Motion



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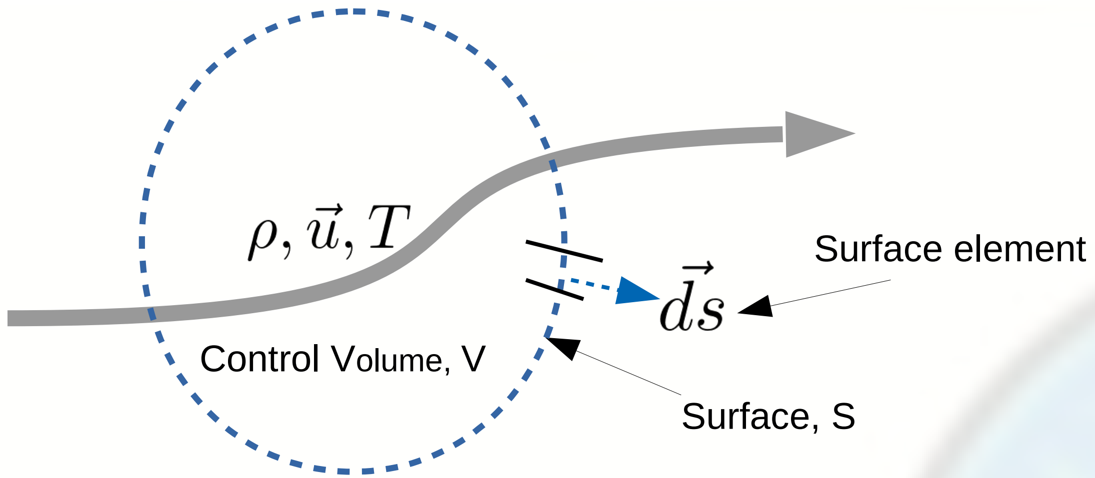
… + equation of state, equations for viscous stresses …

Total energy per unit volume, e(T)

Equations of Fluid Motion



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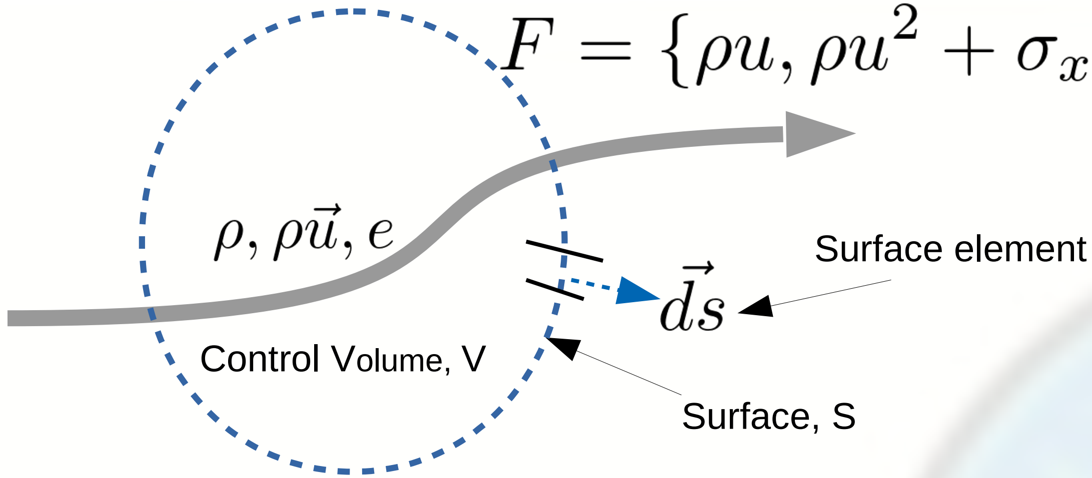
Equations of Fluid Motion

$$F = \{\rho u, \rho u^2 + \sigma_x, (e + \sigma_x)u + \ldots)\}^T$$

$$U = \{\rho, \rho\vec{u}, e\}^T$$



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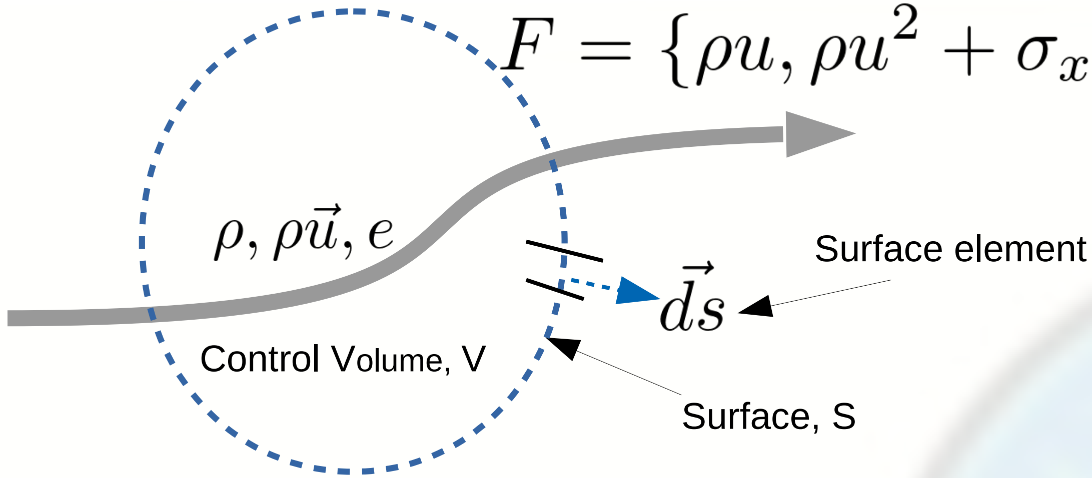

Conservation Equations: conservation law/divergence law form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = Q$$

Equations of Fluid Motion

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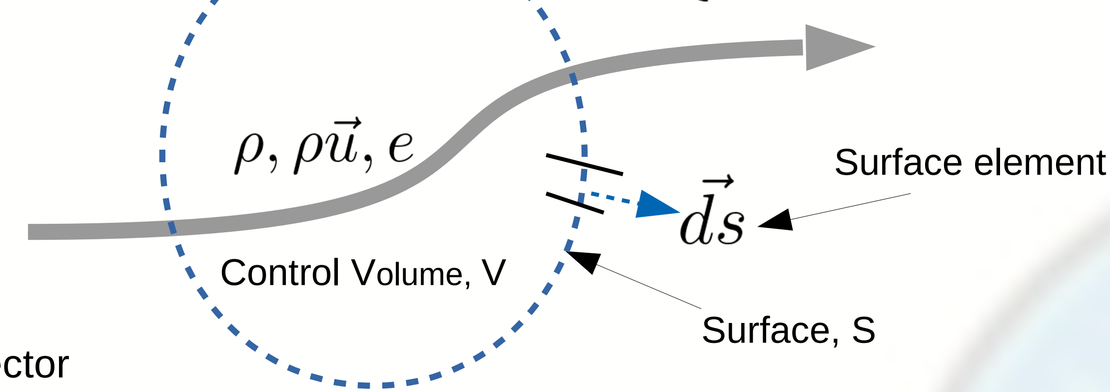

State vector

Flux vectors

Conservation Equations: conservation law/divergence law form

Source term

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Equations of Fluid Motion

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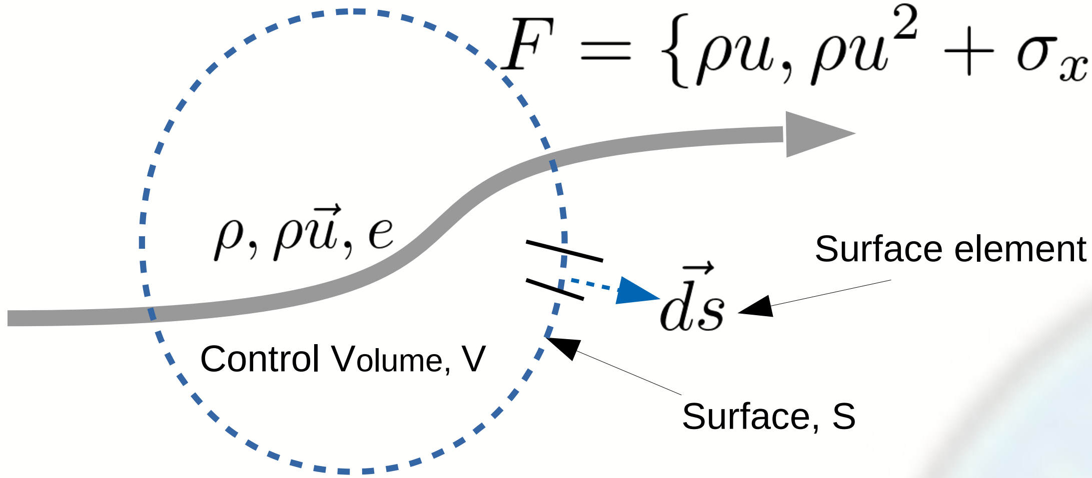

Conservation Equations: integral form

$$\int_V \frac{\partial U}{\partial t} dV + \int_V \vec{\nabla} \cdot \vec{F} dV = \int_V Q dV$$

Equations of Fluid Motion

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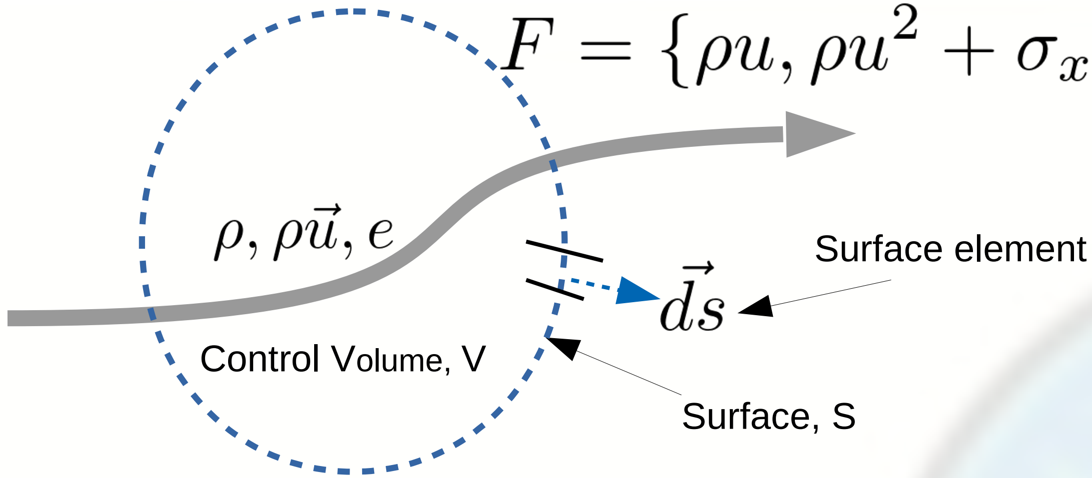
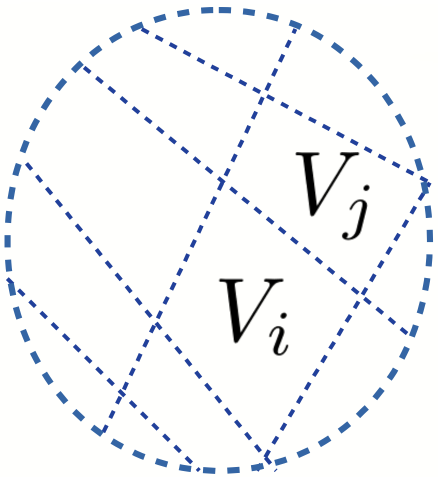


Conservation Equations: integral form
… apply Gauss' Theorem

$$\int_V \frac{\partial U}{\partial t} dV + \frac{1}{V}\int_S \vec{F}\cdot\vec{ds} - \frac{1}{V}\int Q dV$$

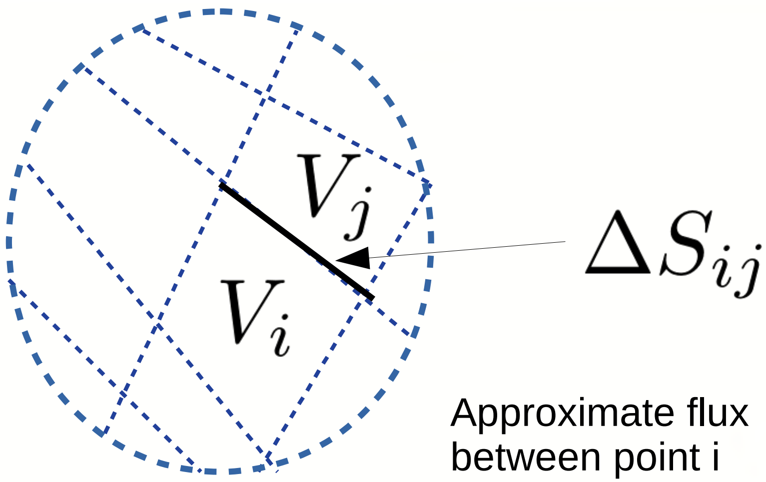
Equations of Fluid Motion



Discretization …

$$\int_V \frac{\partial U}{\partial t} dV + \frac{1}{V}\int_S \vec{F}\cdot\vec{ds} - \frac{1}{V}\int Q dV$$

Equations of Fluid Motion



Approximate flux between point i and j

Surface area

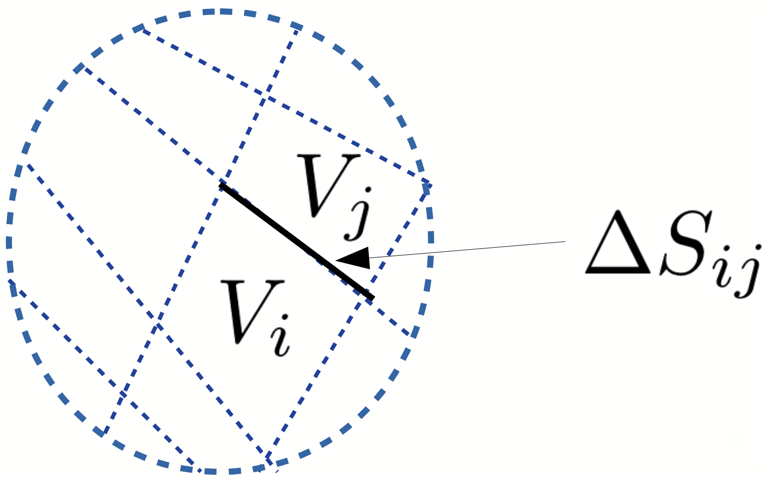
Numerical residual
In SU2:
Res_Flow[0]

Discretization:

$$\int_{V_i} \frac{\partial U}{\partial t} dV + \sum_{j \in N(i)} \tilde{F}_{ij} \Delta S_{ij} - Q|V_i| = \int_{V_i} \frac{\partial U}{\partial t} dV + R_i(U) = 0$$

Each of these terms will be expressed in terms of the values at vertices of the mesh. Several methods exist for how to approximate the flux vectors and gradients

Equations of Fluid Motion



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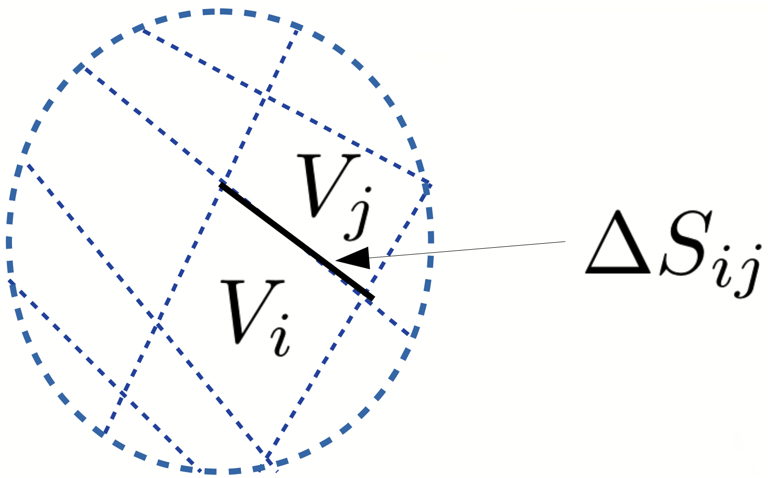
Iterations of the solution for the state vector U (finite volume formulation):

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{V_{i,j}} f(\vec{F}_{i,j}^n, \Delta S_{i,j}) \quad j \in N(i)$$

Equations of Fluid Motion



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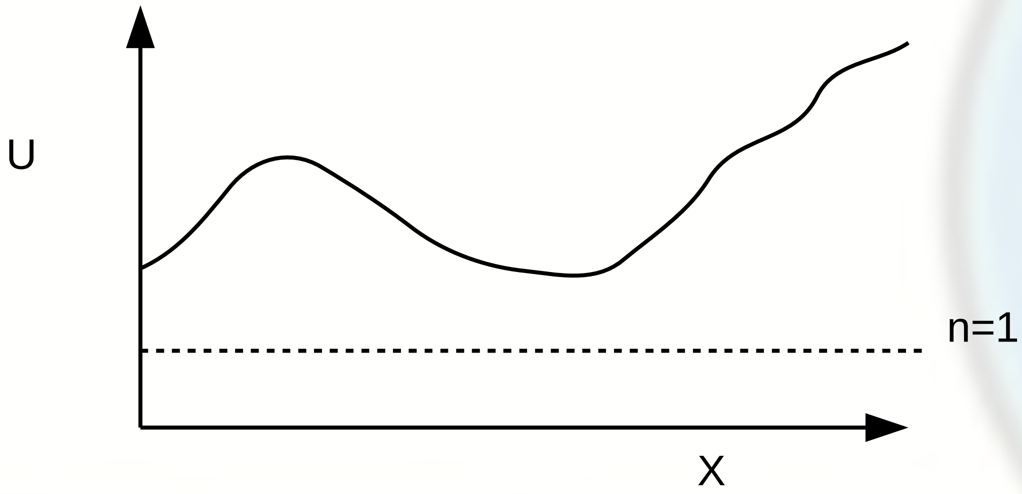
Iterations of the solution for the state vector U (finite difference formulation):

$$U_i^{n+1} = U_i^n - \Delta t \left(\frac{D \cdot F_{i,j}^n}{\Delta x} + \frac{D \cdot G_{i,j}^n}{\Delta y} \right), \; j \in N(i)$$

Numerical Methods

- Start from some initial guess of the solution U_i at the points in the mesh.
- Update the value of U_i based on approximations to the flux vectors between i and all its neighbors j.
- Continue until the residual approaches 0.
- How well flux is approximated, and how quickly the residual will approach 0 depends on the choice of numerical methods, the mesh refinement, and other options.



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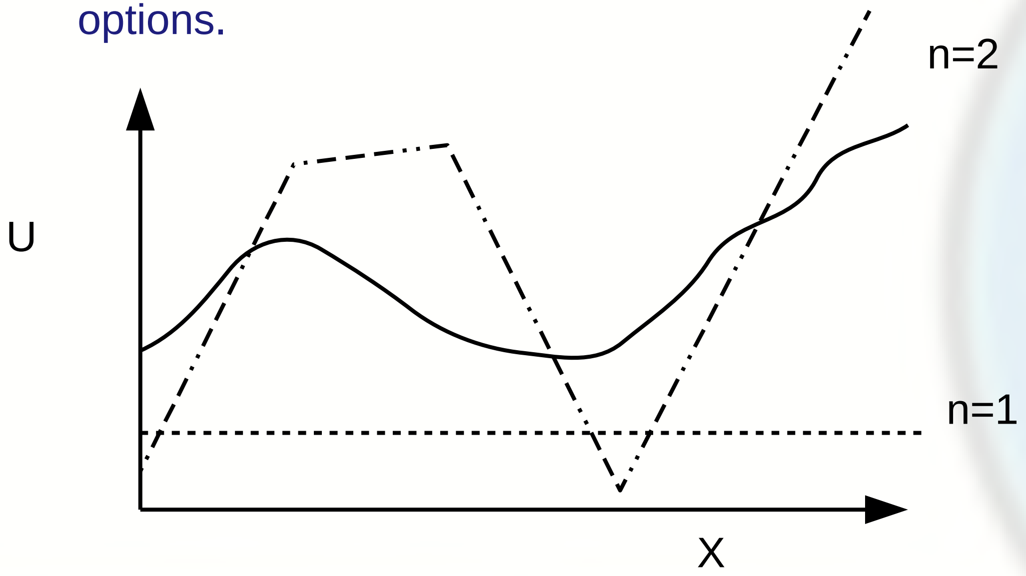

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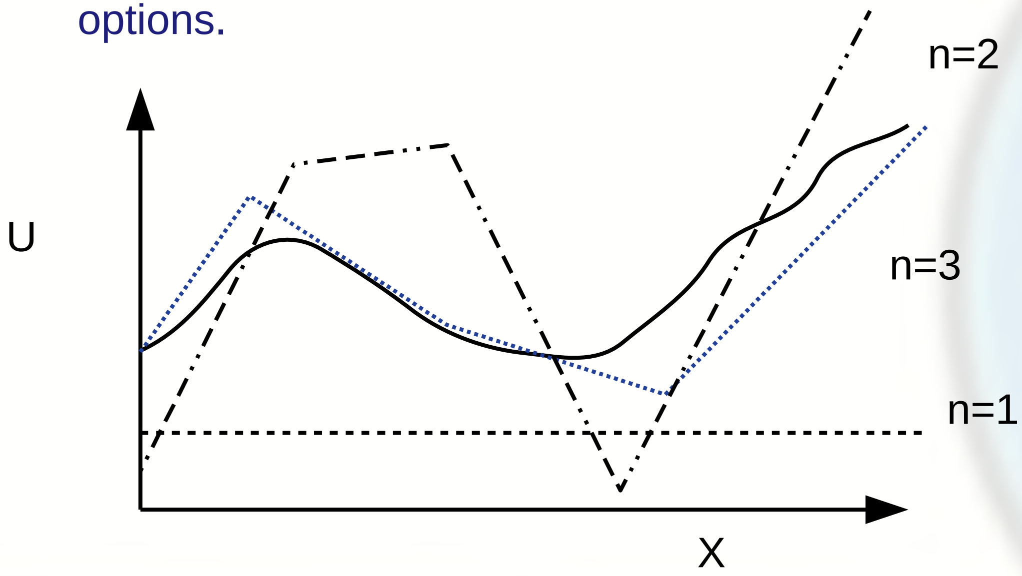

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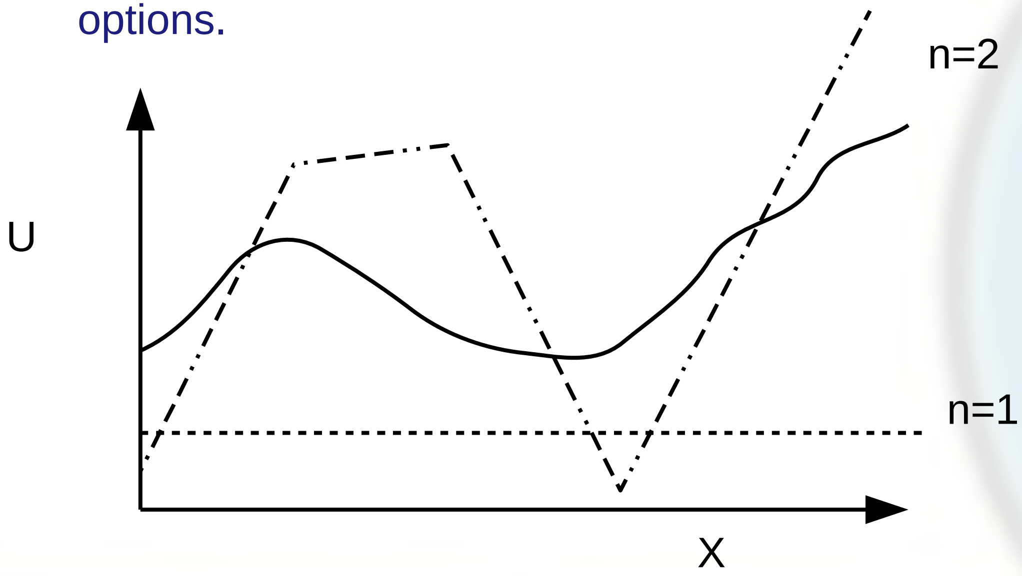

What happens when the solution diverges?

Numerical Methods

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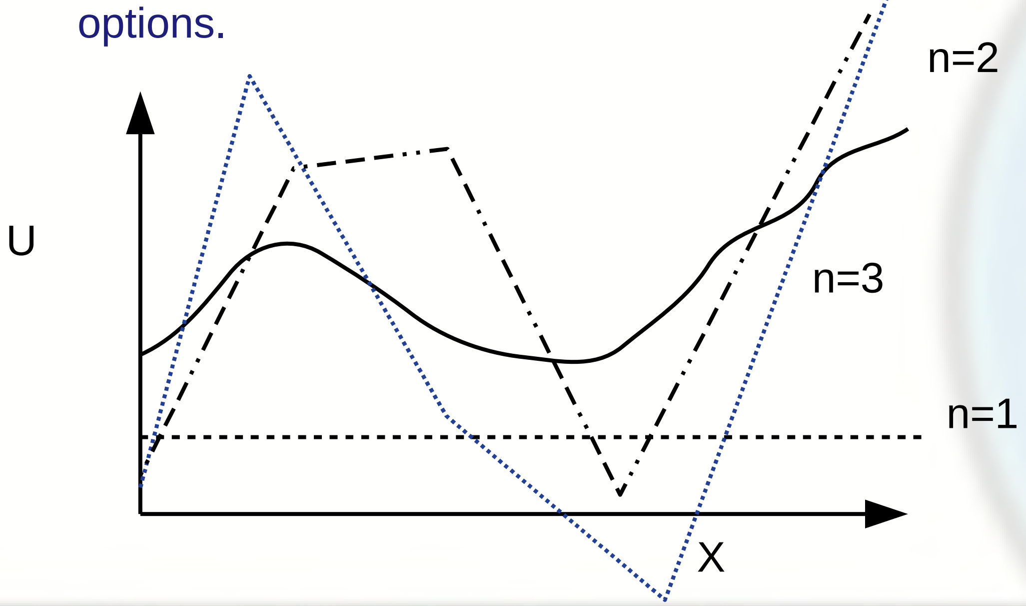

What happens when the solution diverges?

The CFL Number

- The Courant, Friedrichs and Lewy (CFL) number is a condition of stability for explicit time-backwards in space difference method.
  - Explicit in time: depends only on the solution at time n
  - Implicit in space: depends on the solution at multiple locations i
  - Derivation from von Neumann stability analysis applied to numerical algorithms. For more detail, see textbooks on CFD and numerical methods.
  - CFL = $\left|\frac{c\,\Delta t}{\Delta x}\right|$

Index in time

$$u_i^{n+1} = u_i^n - \frac{c\,\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

Index in space

Substituting a Fourier component, one can find the method is stable for CFL ≤ 1

- For time-implicit methods, CFL does not need to be less than 1 to be stable – no strict limit, dependent on the problem being solved and the numerical methods chosen.
- ΔX is controlled by the mesh

The CFL Number

Controlled by:
CFL_NUMBER, mesh, …

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Meshes

In SU2, a primal-dual mesh is used, which constructs control volumes based on connecting the midpoints and centroids of all the edges and faces of the cells of the initial grid. This allows fluxes to be computed over the edges defined in the primal grid.

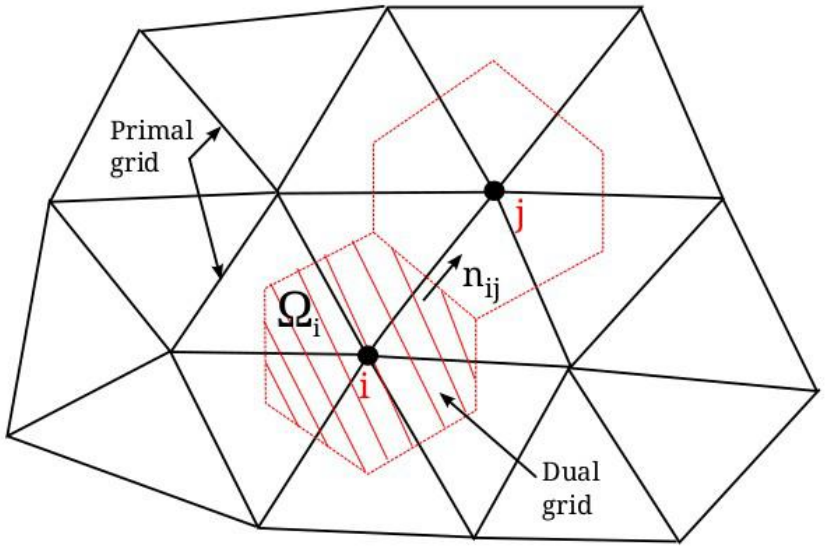

Image credit: R. Sanchez, R. Palacios, T.D. Economon, H.L. Kline, J.J. Alonso, and F. Palacios. Towards a fluid-structure interaction solver for problems with large deformations within the open-source SU2 suite. In 57th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, 2016

Numerical Methods: Related Vocabulary

- Examples: Roe, JST, Lax-Friedrich, CUSP, AUSM, …
  - Described in detail in many references.
  - Actual implementation may vary between flow solvers.
- **Difference operators**/ derivative approximations: what points do you use for the Taylor Series, and what order terms do you keep? Forward difference, backwards difference, central difference, higher order...
- **Truncation error**: how large were the terms you dropped from the Taylor series?
- **Dissipation**: how much will the even-ordered neglected Taylor series terms round out sharp features of the flow?
- **Flux-splitting schemes**: do you use a different difference operator depending on what direction the information is moving.
- **PDE classification** for fluid flows: **Elliptic** where locally subsonic, **Hyperbolic** where locally supersonic.
  - Elliptic: characteristics go in different directions, smooth solutions.
  - Hyperbolic: characteristics go at different speeds in the same direction, solutions can have discontinuities.

Numerical Methods: Related Vocabulary

Controlled by:
[FLOW, FEM] NUMERICAL METHOD DEFINITION
section in the SU2 config file

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Conclusions

Further Reading:

- Numerical methods:
  - Various textbooks by Anderson, Hoffman & Chiang, Thompson
  - Graduate level CFD coursework
  - Journal papers
- SU2 settings: appropriate numerical methods and CFL numbers
  - Tutorial files & test cases
- More help:
  - Cfd-online.com: wiki and forums
  - Turbulence modeling resource

Acknowledgments

- Professor Robert MacCormack's course on Numerical Computation of Compressible Viscous Flow deserves much of the credit for the content in this presentation.